Local Existence and Continuation Criterion for Solutions of the Spherically Symmetric Einstein - Vlasov - Maxwell System

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Abstract

Using the iterative scheme we prove the local existence and uniqueness of solutions of the spherically symmetric Einstein-Vlasov-Maxwell system with small initial data. We prove a continuation criterion to global in-time solutions.

1   Introduction

In [5], the authors prove the global existence solutions of spatial asymptotically flat spherically symmetric Einstein-Vlasov system. This provides a base for the mathematical study of gravitational collapse of collisionless matter; for related works see [2], [4], [7], [8]. That study concerns uncharged particles. We consider, under the same assumption of spherical symmetry, the case where the particles are charged. To describe the full physical situation, we must then couple the previous system to the Maxwell equations that determine the electromagnetic field created by the fast moving charged particles, and that reduces, in the spherically symmetric case, to its electric part.

It is appropriate at this point to examine the motivation for considering this particular problem which unlike the probleme [5] has no direct astrophysical applications, there are, however, two reasons why the problem is interesting. The first reason is that it extends the knowledge of the Cauchy problem for systems involving the Vlasov equation (which models collisionless matter) and it will be seen that it gives rise to new mathematical features compared to those cases studied up to now. The second reason is connected with the fact that it would be desirable to extend the work of [5] beyond spherically symmetric. In particular, it would be desirable from a physical point of view to include the phenomenon of rotation. Unfortunately, presently available techniques do not suffice to get away from spherical symmetry. In this situation it is possible to attempt to obtain further intuition by using the analogy between angular

1

momentum and charge, summed up in John Wheeler's statements, "charge is poor man's angular momentum". Thus we study spherical systems with charge in hope that this will give us insight into non-spherical systems without charge. This strategy has recently been pursued in the case of a scalar field as matter model, with interesting results [1].

Due to the presence of electromagnetic field, the matter quantities are not compactly supported in the spatial variable as it is the case for uncharged particles. So, with this default of compactness, it becomes difficult to prove that the sequence of iterates we use is well defined and converges to a unique solution of the Cauchy problem. The interest of this work lies on the fact that, with a weak regularity condition on matter quantities, the authors prove a local existence theorem and a continuation criterion for solutions which may allow to study the global behaviour of such solutions. We are not aware that this has been done before.

In our specific case, we are led to a difficulty in solving the Cauchy problem by following [5]. Let us first recall the situation in [5] before seeing how it changes in the case of charged particles. In [5], using the assumption of spherical symmetry, the authors look for two metrics functions λ and μ, that depend only on the time coordinate t and the radial coordinate r, and for a distribution function f of the uncharged particles that depends on t, r and on the 3-velocity v of the particles; the metric functions λ, μ are subject to the Einstein equations with sources generated by the distribution function f of the collisionless uncharged particles which is itself subject to the Vlasov equation. They show that the Einstein equations to determine the unknown metric functions λ and μ, turn out to be two first order O.D.E. in the radial variable r, coupled to the Vlasov equation in f. Putting $t = 0$, and denoting by $\overset{\circ}{\lambda}(r)$, $\overset{\circ}{\mu}(r)$ and $\overset{\circ}{f}(r,v)$ the initial data for $\lambda(t,r)$, $\mu(t,r)$ and $f(t,r,v)$ respectively, the constraints equations on the initial data can be solved easily and they need just to prescribe an appropriate condition on $\overset{\circ}{f}(r,v)$ to obtain a unique local solution of the Cauchy problem.

In the case of charged particles, due to the presence of the electromagnetic field in the source terms of Einstein's equations, the initial value problem is not easy to solve. We consider the case of a spherically symmetric electric field $\vec{E}$ of the form $\vec{E}(t,r) = e(t,r)\frac{\vec{r}}{r}$, where $e(t,r)$ is an unknown scalar function and $\vec{r}$ the position vector in $\mathbb{R}^3$. We denote by $\overset{\circ}{e}$ the initial datum for $e(t,r)$. The Einstein - Maxwell equations imply three constraints equations on the initial data, that are a singular first order O.D.E in the radial variable r. In [3], using singular O.D.E techniques, the authors describe one large class of functions $\overset{\circ}{f}$ for which the constraints equations on the initial data are solved. In this paper, we use the above result to define the iterates and we obtain sequences of iterates that converge to the unique local solution of the initial value problem. Moreover we prove the continuation criterion, i.e the control of momenta in supp f which may allow the extendability of the solution for all time, proving the extension of the results of [5] to the case of charged particles with the above indicated

2

consequences.

The paper is organized as follows. In Sect. 2, we recall the general formulation of the Einstein-Vlasov-Maxwell system, from which we deduce the relevant equations in the spherically symmetric spatial asymptotically flat case. In Sect. 3, we establish some properties of the characteristics of the Vlasov equation and we show how to solve each equation when the others unknown are given. In Sect.4, we prove a local existence and uniqueness theorem of solutions for the system, together with a continuation criterion for such solutions.

2   Derivation of the relevant equations

We consider fast moving collisionless particles with charge q. The basic spacetime is $(\mathbb{R}^4, g)$, with g a Lorentzian metric with signature $(-, +, +, +)$. In what follows, we assume that Greek indices run from 0 to 3 and Latin indices from 1 to 3, unless otherwise specified. We also adopt the Einstein summation convention. The metric g reads locally, in cartesian coordinates $(x^\alpha) = (x^o, x^i) \equiv (t, \tilde{x})$:

$$ds^2 = g_{\alpha\beta} dx^\alpha \otimes dx^\beta \tag{2.1}$$

The assumption of spherical symmetry means that we can take g of the following form (Schwarzschild coordinates)[10]

$$ds^2 = -e^{2\mu} dt^2 + e^{2\lambda} dr^2 + r^2 (d\theta^2 + (\sin\theta)^2 d\varphi^2) \tag{2.2}$$

where $\mu = \mu(t, r)$; $\lambda = \lambda(t, r)$; $t \in \mathbb{R}$; $r \in [0, +\infty[$; $\theta \in [0, \pi]$; $\varphi \in [0, 2\pi]$. The Einstein - Vlasov - Maxwell system can be written:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi (T_{\alpha\beta}(f) + \tau_{\alpha\beta}(F)) \tag{2.3}$$

$$\mathcal{L}_{X(F)} f = 0 \tag{2.4}$$

$$\nabla_\alpha F^{\alpha\beta} = J^\beta; \quad \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0 \tag{2.5}$$

with:

$$T_{\alpha\beta}(f) = -\int_{\mathbb{R}^3} p_\alpha p_\beta f \omega_p; \quad \tau_{\alpha\beta}(F) = -\frac{g_{\alpha\beta}}{4} F_{\gamma\nu} F^{\gamma\nu} + F_{\beta\gamma} F_\alpha{}^\gamma$$

$$J^\beta(f)(x) = q \int_{\mathbb{R}^3} p^\beta f(x, p) \omega_p, \quad \omega_p = |g|^{\frac{1}{2}} \frac{dp^1 dp^2 dp^3}{p_0}, \quad p_0 = g_{00} p^0,$$

$$X^\alpha(F) = (p^\alpha, -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma - q p^\beta F_\beta{}^\alpha),$$

where $\Gamma^\alpha_{\beta\gamma}$ denote the Christoffel symbols, and $\mathcal{L}_{X(F)}$ the Lie derivative. Here, $x = (x^\alpha)$ is the position and $p = (p^\alpha)$ is the 4-momentum of the particles. In the expressions above, f stands for the distribution function of the charged particles,

F stands for the electromagnetic field created by the charged particles. Here (2.3) are the Einstein equations for the metric tensor $g = (g_{\alpha\beta})$ with sources generated by both f and F, that appear in the stress-energy tensor $8\pi(T_{\alpha\beta} + \tau_{\alpha\beta})$. Equation (2.4) is the Vlasov equation for the distribution function f of the collisionless particles and (2.5) are the Maxwell equations for the electromagnetic field F, with source(current) generated by f through $J = J(f)$. One verifies (using the normal coordinates) that the conservation laws $\nabla_\alpha(T^{\alpha\beta} + \tau^{\alpha\beta}) = 0$ hold if f satisfies the Vlasov equation.

By the assumption of spherical symmetry, we can take g in the form (2.2). One shows, using the Maxwell equation that F reduces to its electric part, we take it in the form $E = (E^\alpha)$ with $E^0 = 0$, $E^i = e(t,r)\frac{x^i}{r}$, and then, a straightforward calculation shows that:

$$\tau_{00} = \frac{1}{2}e^{2(\lambda+\mu)}e^2(t,r); \quad \tau_{0i} = 0$$

$$\tau_{ij} = \frac{1}{2}e^{2\lambda}e^2(t,r)\{(\delta_{ij} - \frac{x_i x_j}{r^2}) - e^{2\lambda}\frac{x_i x_j}{r^2}\},$$

where δ_{ij} denote the Kronecker symbols.

These relations and results of [5] show that the spherically symmetric Einstein - Vlasov - Maxwell system writes as the following system in λ, μ, f, e:

$$e^{-2\lambda}(2r\lambda' - 1) + 1 = 8\pi r^2 \rho \tag{2.6}$$

$$\dot{\lambda} = -4\pi r e^{\lambda+\mu} k \tag{2.7}$$

$$e^{-2\lambda}(2r\mu' - 1) + 1 = 8\pi r^2 p \tag{2.8}$$

$$e^{-2\lambda}(\mu'' + (\mu' - \lambda')(\mu' + \frac{1}{r})) - e^{-2\mu}(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu})) = 4\pi\bar{q} \tag{2.9}$$

$$\frac{\partial f}{\partial t} + e^{\mu-\lambda}\frac{v}{\sqrt{1+v^2}}\cdot\frac{\partial f}{\partial\tilde{x}} - \left(e^{\mu-\lambda}\mu'\sqrt{1+v^2} + \dot{\lambda}\frac{\tilde{x}.v}{r} - qe^{\lambda+\mu}e(t,r)\right)\frac{\tilde{x}}{r}\cdot\frac{\partial f}{\partial v} = 0 \tag{2.10}$$

$$\frac{\partial}{\partial r}(r^2 e^\lambda e(t,r)) = qr^2 e^\lambda M \tag{2.11}$$

$$\frac{\partial}{\partial t}(e^\lambda e(t,r)) = -\frac{q}{r}e^\mu N \tag{2.12}$$

where $\lambda' = \frac{\partial\lambda}{\partial r}$; $\dot{\lambda} = \frac{\partial\lambda}{\partial t}$ and:

$$\rho(t,\tilde{x}) = \int_{\mathbb{R}^3} f(t,\tilde{x},v)\sqrt{1+v^2}dv + \frac{1}{2}e^{2\lambda(t,\tilde{x})}e^2(t,\tilde{x}) \tag{2.13}$$

4

$$k(t, \tilde{x}) = \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r} f(t, \tilde{x}, v) dv \tag{2.14}$$

$$p(t, \tilde{x}) = \int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r}\right)^2 f(t, \tilde{x}, v) \frac{dv}{\sqrt{1+v^2}} - \frac{1}{2} e^{2\lambda(t,\tilde{x})} e^2(t, \tilde{x}) \tag{2.15}$$

$$\bar{q}(t, \tilde{x}) = \int_{\mathbb{R}^3} \left(v^2 - \left(\frac{\tilde{x}.v}{r}\right)^2\right) f(t, \tilde{x}, v) \frac{dv}{\sqrt{1+v^2}} + e^{2\lambda(t,\tilde{x})} e^2(t, \tilde{x}) \tag{2.16}$$

$$M(t, \tilde{x}) = \int_{\mathbb{R}^3} f(t, \tilde{x}, v) dv; \quad N(t, \tilde{x}) = \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{\sqrt{1+v^2}} f(t, \tilde{x}, v) dv. \tag{2.17}$$

Here (2.6), (2.7), (2.8) and (2.9) are the Einstein equations for λ and μ, (2.10) is the Vlasov equation for f, (2.11) and (2.12) are the Maxwell equations for e. Here $\tilde{x}$ and v belong to $\mathbb{R}^3$, $r := |\tilde{x}|$, $\tilde{x}.v$ denotes the usual scalar product of vectors in $\mathbb{R}^3$, and $v^2 := v.v$. The distribution function f is assumed to be invariant under simultaneous rotations of $\tilde{x}$ and v, hence ρ, k, p, M and N can be regarded as functions of t and r. It is assumed that $f(t)$ has compact support for each fixed t. We are interested in spatial asymptotically flat space-time with a regular center, which leads to the boundary conditions that:

$$\lim_{r \to \infty} \lambda(t, r) = \lim_{r \to \infty} \mu(t, r) = \lim_{r \to \infty} e(t, r) = \lambda(t, 0) = e(t, 0) = 0 \tag{2.18}$$

Now, define the initial data by:

$$\begin{cases} f(0, \tilde{x}, v) = \overset{\circ}{f}(\tilde{x}, v); \quad \lambda(0, \tilde{x}) = \overset{\circ}{\lambda}(\tilde{x}) = \overset{\circ}{\lambda}(r) \\ \mu(0, \tilde{x}) = \overset{\circ}{\mu}(\tilde{x}) = \overset{\circ}{\mu}(r); \quad e(0, \tilde{x}) = \overset{\circ}{e}(\tilde{x}) = \overset{\circ}{e}(r) \end{cases} \tag{2.19}$$

with $\overset{\circ}{f}$ being a C^∞ function with compact support, which is nonnegative and spherically symmetric, i.e

$$\forall A \in SO(3), \forall (\tilde{x}, v) \in \mathbb{R}^6, \overset{\circ}{f}(A\tilde{x}, Av) = \overset{\circ}{f}(\tilde{x}, v).$$

We have to solve the boundary initial value problem (2.6), (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), (2.18), and (2.19).

3 Preliminary results, conservation laws, and reduced systems

For given ρ and p, (2.6) and (2.8) determine (λ, μ) and the right hand side of (2.7) is known when k is given. So we can use this equation to define $\tilde{\lambda}$ as the time derivative of λ. Indeed we show later that $\tilde{\lambda} = \dot{\lambda}$. Therefore, we call auxiliary system to equations (2.7) and (2.10) the following equations:

$$\frac{\partial f}{\partial t} + e^{\mu-\lambda} \frac{v}{\sqrt{1+v^2}} \cdot \frac{\partial f}{\partial \tilde{x}} - \left(e^{\mu-\lambda} \mu' \sqrt{1+v^2} + \tilde{\lambda} \frac{\tilde{x}.v}{r} - q e^{\lambda+\mu} e(t, r)\right) \frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial v} = 0 \tag{3.1}$$

where
$$\tilde{\lambda} = -4\pi e^{\lambda+\mu} k \qquad (3.2)$$

together with (2.6), (2.8) and (2.11). It appears clearly that a solution (λ, μ, e, f) of the coupled system (2.6), (2.8), (2.11), and (3.1) that satisfies (2.9), (2.10), (2.12) and $\tilde{\lambda} = \dot{\lambda}$ is a solution of the full initial system. Before we do so, we make precise the regularity properties which we require of solution:

Definition Let $I \subset \mathbb{R}$ be an interval.

a) $f : I \times \mathbb{R}^6 \to \mathbb{R}^+$ is regular, if $f \in C^1(I \times \mathbb{R}^6)$, $f(t)$ is spherically symmetric and $\mathrm{supp} f(t)$ is compact for all $t \in I$.

b) $\rho(\text{or } p, \bar{q}) : I \times \mathbb{R}^3 \to \mathbb{R}$ is regular, if $\rho \in C^1(I \times \mathbb{R}^3)$, $\rho(t)$ is spherically symmetric for all $t \in I$.

c) $M(\text{or } N) : I \times \mathbb{R}^3 \to \mathbb{R}$ is regular, if $M \in C^1(I \times \mathbb{R}^3)$, $M(t)$ is spherically symmetric and $\mathrm{supp} M(t)$ is compact for all $t \in I$.

d) $k : I \times \mathbb{R}^3 \to \mathbb{R}$ is regular, if $k \in C(I \times \mathbb{R}^3) \cap C^1(I \times \mathbb{R}^3 \setminus \{0\})$, $k(t)$ is spherically symmetric, $\mathrm{supp} k(t)$ compact and $k(t) \in C^1([0, +\infty[)$ for all $t \in I$.

e) $\lambda, \mu, \tilde{\lambda} : I \times [0, +\infty[\to \mathbb{R}$ is regular, if $\lambda, \mu, \tilde{\lambda} \in C^2(I \times [0, +\infty[)$; $\lambda, \mu, \tilde{\lambda}$ satisfy (2.18) and
$$\dot{\lambda}(t, 0) = \lambda'(t, 0) = \mu'(t, 0) = \tilde{\lambda}'(t, 0) = 0,$$
for all $t \in I$.

f) $e : I \times [0, +\infty[\to \mathbb{R}$ is regular if $e, e' \in C(I \times [0, +\infty[)$ and e satisfies (2.18).

Remark 3.1 *If f and e is regular then quantities ρ, p, k, $\bar{q}$, M and N defined from f are also regular in the appropriate sense.*

Let us now consider the Vlasov equation (3.1) for prescribed functions λ, μ, $\tilde{\lambda}$ and e.

Proposition 3.1 *Let $I \in \mathbb{R}$ be an interval with $0 \in I$, λ, μ, $\tilde{\lambda}$ and e regular on $I \times [0, +\infty[$, with $\lambda \geq 0$, $\mu \leq 0$ and define*

$$F_1(s, \tilde{x}, v) = e^{\mu-\lambda} \frac{v}{\sqrt{1+v^2}}$$

$$F_2(s, \tilde{x}, v) = \begin{cases} -\left(\tilde{\lambda} \frac{\tilde{x}.v}{r} + e^{\mu-\lambda}\mu'\sqrt{1+v^2} - qe^{\mu+\lambda}e \right) \frac{\tilde{x}}{r} & \text{if} \quad \tilde{x}, v \in \mathbb{R}^3, \tilde{x} \neq 0 \\ 0 & \text{if} \quad \tilde{x} = 0, \quad v \in \mathbb{R}^3 \end{cases}$$

and

$$F(s, z) = F(s, \tilde{x}, v) = (F_1, F_2)(s, \tilde{x}, v); \quad s \in I \quad z = (\tilde{x}, v) \in \mathbb{R}^6.$$

Then

a) $F \in C^1(I \times \mathbb{R}^6)$.

b) For every $t \in I$, $z \in \mathbb{R}^6$, the characteristics system

$$\dot{z} = F(s, z)$$

has a unique solution $s \mapsto Z(s, t, z) = (X, V)(s, t, z)$ with $Z(t, t, z) = z$. Moreover, $Z \in C^1(I^2 \times \mathbb{R}^6)$ is a C^1-diffeomorphism of $\mathbb{R}^6$ with inverse $Z(t, s, .)$, $s, t \in I$, and

$$(X, V)(s, t, A\tilde{x}, Av) = (AX, AV)(s, t, \tilde{x}, v)$$

for $A \in SO(3)$ and $\tilde{x}, v \in \mathbb{R}^3$.

c) For a nonnegative, spherically symmetric function $\overset{\circ}{f} \in C_c^1$,

$$f(t, z) = f(t, \tilde{x}, v) = \overset{\circ}{f}(Z(0, t, z)) = \overset{\circ}{f}(x^i(t, z), v^i(t, z))$$

$t \in I$, $\tilde{x}, v \in \mathbb{R}^3$, defines the unique regular solution of (3.1) with $f(0) = \overset{\circ}{f}$.

d) If f is the regular solution of (2.10), then

$$\frac{\partial}{\partial t}\left(e^\lambda \int_{\mathbb{R}^3} f \, dv\right) + \underset{\tilde{x}}{\mathrm{div}}\left(e^\mu \int_{\mathbb{R}^3} \frac{v}{\sqrt{1+v^2}} f \, dv\right) = 0 \qquad (3.3)$$

where $\underset{\tilde{x}}{\mathrm{div}}$ is divergence in the Euclidian metric on $\mathbb{R}^3$ and thus the quantity

$$\int\int_{\mathbb{R}^6} e^{\lambda(t, \tilde{x})} f(t, \tilde{x}, v) \, d\tilde{x} dv, \quad t \in I \qquad (3.4)$$

is conserved.

Proof: The crucial point in the proof of part a) is the regularity of F_2 at $r = 0$. Now the term

$$\frac{x^i}{r} \mu'(s, r) = \frac{\partial \mu}{\partial x^i}(s, r)$$

is continuously differentiable with respect to $\tilde{x} \in \mathbb{R}^3$ and vanishes at $r = 0$ by virtue of the regularity of μ. The term $\tilde{\lambda}\frac{x_i x_j}{r^2}$ is continuously differentiable with respect to $\tilde{x}$, using the regularity of $\tilde{\lambda}$. The continuously differentiability of both terms with respect to t at $\tilde{x} = 0$ follows from the fact that

$$\dot{\tilde{\lambda}}(t, 0) = \dot{\mu}'(t, 0) = 0, \quad t \in I$$

and the following expression

$$\frac{\partial}{\partial x^k}\left(e^{\lambda+\mu} e \frac{x^i}{r}\right) = e^{\lambda+\mu}\left(e(\lambda' + \mu')\frac{x^i x_k}{r^2} + e'\frac{x^i x_k}{r^2} + \frac{e}{r}\delta_k^i - e\frac{x^i x_k}{r^3}\right)$$

shows that the term $e\frac{\tilde{x}}{r}$ is also continuously differentiable at $r = 0$, since by the regularity of e, we have:

$$e(t, r) = re'(t, 0) + r\varepsilon(t, r), \quad \lim_{r \to 0} \varepsilon(t, r) = 0.$$

Therefore F_2 is continuously differentiable on $I \times \mathbb{R}^6$. This implies local existence, uniqueness and regularity of $Z(., t, z)$. Since

$$\mid \dot{x} \mid = \left| \frac{dx}{ds} \right| = \left| e^{\mu - \lambda} \frac{v}{\sqrt{1 + v^2}} \right| \le e^{\mu - \lambda} \le 1$$

$X(., t, z)$ remains bounded on bounded sub-intervals of I. On the other hand, by regularity of λ, μ, e and

$$\mid \dot{v} \mid \le \mid \tilde{\lambda} \mid \mid v \mid + \mid \mu' \mid (1 + \mid v \mid) + \mid q \mid \mid e \mid e^{\lambda + \mu}$$

which is bounded on every bounded sub-interval of I by the Gronwall lemma, the same is true for $V(., t, z)$. Therefore, $Z(., t, z)$ exists on I. The other assertions in b) are standard, or follow by uniqueness. Assertion c) is an immediate consequence of b) and the fact that according to (2.10), f remains constant along the trajectories. Now, to prove part d), we multiply (2.10) with e^λ, integrate with respect to v and apply Gauss theorem to obtain (3.4). The conservation law in d) corresponds to conservation of number of particles. The term e^λ comes from the fact that the coordinates v on the mass shell are not the canonical momenta corresponding to $\tilde{x}$, and proposition 3.1 is proved.

We need the following result obtained by a direct computation to control certain derivatives of the unknown [6]:

Lemma 3.1 *Let $I \in \mathbb{R}$ be an interval, let λ, μ, $\tilde{\lambda}$, $e : I \times [0, +\infty[\to$ be regular, and define $(X, V)(s) = (X, V)(s, t, z)$ for $(s, t, z) \in I^2 \times \mathbb{R}^6$ as in proposition 2.1. For $j \in \{1, ..., 6\}$ define*

$$\xi_j(s) = \frac{\partial X}{\partial z_j}(s, t, z)$$

$$\eta_j(s) = \frac{\partial V}{\partial z_j}(s, t, z) + \sqrt{1 + V^2(s)} e^{(\lambda - \mu)(s, X(s))} \tilde{\lambda}(s, X(s)) \frac{X(s)}{\mid X(s) \mid} \frac{X(s)}{\mid X(s) \mid} \cdot \frac{X}{z_j}(s, t, z).$$

Then,

$$\frac{d\xi_j}{ds} = a_1(s, X(s), V(s))\xi_j + a_2(s, X(s), V(s))\eta_j$$

$$\frac{d\eta_j}{ds} = (a_3 + a_5)(s, X(s), V(s))\xi_j + a_4(s, X(s), V(s))\eta_j$$

where the coefficients of matrices $a_1, ..., a_5$ are a regular functions of λ, μ, e and their derivatives.

Note that a_1 , a_2, a_4 and a_5 are the same as in ([6], lemma 2.3). But here, due to the presence of electromagnetic field, coefficients $(a_3(s, \tilde{x}, v))^i_k$ of the matrix

a_3 contents the corresponding terms in [6] plus an additional one that is:

$$q\frac{e}{r}\alpha e^{\lambda}\delta_k^i + q\alpha e^{\lambda}\left(e(\lambda' + \mu') + e' + \frac{e}{r}\tilde{\lambda}\frac{\tilde{x}.v}{\sqrt{1+v^2}} - \frac{e}{r}\right)\frac{x^i x_k}{r^2}$$

Next, we investigate field equations (2.6), (2.8) for given ρ, p and the Maxwell equation (2.11) for given M.

Proposition 3.2 *Let $\bar{\lambda}, \bar{e} : I \times [0, +\infty[\to \mathbb{R}_+$ and $\bar{f} : I \times \mathbb{R}^6 \to \mathbb{R}_+$ be regular and define $\rho = \rho(\bar{f}, \bar{\lambda}, \bar{e})$; $p = p(\bar{f}, \bar{\lambda}, \bar{e})$, $M = M(\bar{f})$ as in (2.13), (2.15) and (2.17), replacing f, λ, e by $\bar{f}$, $\bar{\lambda}$, $\bar{e}$ respectively, and let:*

$$m(t,r) = 4\pi \int_0^r s^2 \rho(t,s)ds = \int_{|y| \leq r} \rho(t,y)dy \tag{3.5}$$

where $t \in I$, $r \in [0, +\infty[$. Then there exists a regular solution (λ, μ, e) of the system (2.6), (2.8) and (2.11) on $I \times [0, +\infty[$ satisfying the boundary conditions (2.18) if and only if:

$$\frac{2m(t,r)}{r} < 1, \quad t \in I, \quad r \in [0, +\infty[. \tag{3.6}$$

The solution is given by

$$e^{-2\lambda(t,r)} = 1 - \frac{2m(t,r)}{r} \tag{3.7}$$

$$\mu'(t,r) = e^{2\lambda(t,r)}\left(\frac{m(t,r)}{r^2} + 4\pi r p(t,r)\right) \tag{3.8}$$

$$\mu(t,r) = -\int_r^{+\infty} \mu'(t,s)ds \tag{3.9}$$

$$\lambda'(t,r) = e^{2\lambda(t,r)}\left(-\frac{m(t,r)}{r^2} + 4\pi r \rho(t,r)\right) \tag{3.10}$$

$$\lambda'(t,r) + \mu'(t,r) = 4\pi e^{2\lambda(t,r)}(\rho(t,r) + p(t,r)) \tag{3.11}$$

$$\lambda(t,r) \geq 0; \quad \mu(t,r) \leq 0; \quad \lambda(t,r) + \mu(t,r) \leq 0 \tag{3.12}$$

and

$$e(t,r) = \frac{q}{r^2}e^{-\lambda(t,r)}\int_0^r s^2 e^{\lambda(t,s)} M(t,s)ds \tag{3.13}$$

for $(t,r) \in I \times [0, +\infty[$.

Proof: First observe that the field equation (2.6) can be written in the form

$$(re^{-2\lambda})' = 1 - 8\pi r^2 \rho$$

which can be integrated on $[0, +\infty[$ subject to the condition $\lambda(t, 0) = 0$ if and only if (3.6) holds, since the equality (3.7) holds only if its right hand side is nonnegative. We obtain (3.8) from (2.8) and using (3.7). So, (3.7), (3.8) and (3.9) clearly define the unique regular solution μ, which due to compact support of $\bar{f}(t)$ converges to 0 for $r \to \infty$. The boundary condition for λ at $r = 0$ follows from the boundedness of ρ at $r = 0$. Now, if we solve (2.6) with unknown λ' and observe (3.7) we obtain (3.10) and (3.8)+(3.10) give (3.11). On the other hand (3.7) gives:

$$\lambda(t, r) = -\frac{1}{2}\text{Log}\left(1 - \frac{2m(t, r)}{r}\right) > 0.$$

Since $1 - \frac{2m(t,r)}{r} < 1$, also $\mu' \geq 0$ and thus $\mu \leq 0$ due to the boundary condition at $r = \infty$. From (3.11) it follows that $\lambda + \mu$ is increasing in r, and since this function vanishes at $r = \infty$, $\lambda + \mu \leq 0$. On the other hand, we obtain (3.13) by integrating (2.11) on $[0, r]$ and using $e(t, 0) = 0$, since $\lambda \geq 0$, λ and M are bounded in r. Now, the differentiability properties of λ, μ and e which are part of definition of being regular are obvious. Then the proof is complete.

We now show that the reduced system mentioned above is equivalent to the full system. We also prove the following conservation law:

$$\frac{\partial \rho}{\partial t} + \underset{\tilde{x}}{\text{div}}\left(e^{\mu - \lambda}\int_{\mathbb{R}^3} vf dv\right) = 0 \tag{3.14}$$

Proposition 3.3 *Let (λ, μ, f, e) be a regular solution of subsystem (2.6), (2.8), (2.10) and (2.11) satisfying the boundary conditions (2.18). Then (λ, μ, f, e) satisfies the full Einstein-Vlasov-Maxwell system (2.6), (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12), and the A.D.M mass*

$$M(t) := \int_{\mathbb{R}^3} \rho(t, y)dy = \lim_{r \to \infty} m(t, r) \tag{3.15}$$

is conserved.

Proof: Using the conservation law (3.3), we can deduce that each solution of equation (2.11) is a solution of (2.12). Also, differentiating the relation (2.13) of ρ with respect to t and using the Vlasov equation (2.10) we obtain, by Gauss theorem:

$$\frac{\partial \rho}{\partial t} = -\underset{\tilde{x}}{\text{div}}\left(e^{\mu - \lambda}\int_{\mathbb{R}^3} vf dv\right) - (\rho + p)(\dot{\lambda} + 4\pi re^{\lambda + \mu}k). \tag{3.14'}$$

So, we will have the conservation law (3.14) if the second term in the right hand side of (3.14') vanishes. But we obtain the latter if (2.7) holds, and this can

10

be established, by differentiating (3.7) with respect to t, using (3.14'), Gauss theorem and the Gronwall lemma. Then, since (3.14) holds, $\frac{dM(t)}{dt} = 0$ and the A.D.M mass is conserved. Next, because (2.7) holds, we use once again the Vlasov equation and Gauss theorem to show that equation (2.9) holds as well and the proof is complete.

Remark 3.2 *We consider the auxiliary system (2.6), (2.8), (2.11), (3.1) and (3.2), which we use in the proof of local existence result in the next section.*

Remark 3.3 *Consider a regular solution $(f, \lambda, \mu, \tilde{\lambda}, e)$ of the auxiliary system. Then, since e is a solution of (2.12), we can conclude that $e \in C^1(I \times \mathbb{R}^3)$ and by the regularity of λ, $g_{\alpha\beta} \in C^1(I \times \mathbb{R}^3)$, where the metric g is given in cartesian coordinates by:*

$$g_{00}(t, \tilde{x}) = -e^{2\mu(t,\tilde{x})}, \ g_{0i}(t, \tilde{x}) = 0, \ g_{ij}(t, \tilde{x}) = \delta_{ij} + (e^{2\lambda(t,\tilde{x})} - 1)\frac{x_i x_j}{r^2}.$$

Proposition 3.4 *Let $(\lambda, \mu, f, e, \tilde{\lambda})$ be a regular solution of (2.6), (2.8), (2.11), (3.1) and (3.2. Then (λ, μ, f, e) solves the full spherically symmetric Einstein-Vlasov-Maxwell system (2.6), (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12).*

Proof: Let $(\lambda, \mu, f, e, \tilde{\lambda})$ be a regular solution of (2.6), (2.8), (2.11), (3.1) and (3.2), by proposition 3.1, we have only to show that $\dot{\lambda} = \tilde{\lambda}$, and this is obtained by differentiating (3.7) w.r.t t, using (3.11) and Gauss theorem. Thus the proof is complete.

Now, we give this result we use later, obtained by induction and integration and that is

Lemma 3.2 *Let $h : [0, t] \to \mathbb{R}$ be a continuous function. Then for all $n \in \mathbb{N}$, $n \geq 1$, we have:*

$$\int_0^t ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 ... \int_0^{s_{n-1}} h(s_n)ds_n = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} h(s)ds$$

(3.16)

We end this section by recalling the constraints equations on the initial data. As the authors show in [3], these equations are:

$$e^{-2\overset{\circ}{\lambda}}(2r\overset{\circ}{\lambda}' - 1) + 1 = 8\pi r^2 \left(\int_{\mathbb{R}^3} \sqrt{1+v^2}\overset{\circ}{f}dv + \frac{1}{2}e^{2\overset{\circ}{\lambda}}\overset{\circ}{e}^2 \right)$$

(3.17)

$$e^{-2\overset{\circ}{\lambda}}(2r\overset{\circ}{\mu}' + 1) - 1 = 8\pi r^2 \left(\int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r}\right)^2 \overset{\circ}{f}\frac{dv}{\sqrt{1+v^2}} - \frac{1}{2}e^{2\overset{\circ}{\lambda}}\overset{\circ}{e}^2 \right)$$

(3.18)

$$\frac{d}{dr}(r^2 e^{\overset{\circ}{\lambda}}\overset{\circ}{e}) = qr^2 e^{\overset{\circ}{\lambda}} \int_{\mathbb{R}^3} \overset{\circ}{f}(r,v)dv$$

(3.19)

4 Local existence and continuation of solutions

In this section we prove a local existence and uniqueness theorem for regular solutions of the initial value problem corresponding to the spatial asymptotically flat, spherically symmetric Einstein-Vlasov-Maxwell system, together with a continuation criterion for such solutions. The basic idea of the proof is to use for given small $\overset{\circ}{f}$, a solution $(\overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{e})$ of the constraints equations (3.17), (3.18) and (3.19) obtained in [3], and proposition 3.2, to construct the iterates and show that these iterates converge to a solution on some interval of the coupled system. Here, compared to the situation met by the authors in [5], the main difficulties are the following: equation (2.6) does not define directly λ for given f as it is the case for Einstein-Vlasov system, and if we consider (2.7) to define λ, then $\dot{\lambda}$ will become very unpleasant to control. The latter difficulty is solved by using the auxiliary system (2.6), (2.8), (2.10), (2.11), (3.1), (3.2) and apply proposition 3.4.

4.1 The construction of iterates

Let $\overset{\circ}{f} \in C^\infty(\mathbb{R}^6)$ be nonnegative, compactly supported and spherically symmetric with

$$8\pi \int_0^r s^2 \left(\int_{\mathbb{R}^3} \overset{\circ}{f}(s,v) \sqrt{1+v^2} dv \right) < r \tag{4.1}$$

Let $\overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{e} \in C^\infty(\mathbb{R}^3)$ be a regular solution of (3.17), (3.18) and (3.19). By proposition 3.4, it sufficient to solve the auxiliary system (2.6), (2.8), (2.10), (2.11), (3.1) and (3.2). Furthermore, it is sufficient to solve this system for $t > 0$, the proof for $t < 0$ would proceed in exactly the same way. Note that by [3], assumption (4.1) on $\overset{\circ}{f}$ ensures the existence of a local solution of the constraints equations, for law charge. We assume that $\mathrm{supp}\overset{\circ}{f} \subset B(r_0) \times B(u_0)$, with $B(r)$ the open ball of $\mathbb{R}^3$, with the center O and the radius r,

$$r_0 = \sup\{|\tilde{x}| \,|\, (\tilde{x}, v) \in \mathrm{supp}\overset{\circ}{f}\} \tag{4.2}$$

$$u_0 = \sup\{|v| \,|\, (\tilde{x}, v) \in \mathrm{supp}\overset{\circ}{f}\}. \tag{4.3}$$

We consider the following iterative scheme:

$$\lambda_0 = \overset{\circ}{\lambda}; \quad \mu_0 = \overset{\circ}{\mu}; \quad f_0 = \overset{\circ}{f}; \quad e_0 = \overset{\circ}{e}; \quad T_0 = +\infty.$$

If $\lambda_{n-1}, \mu_{n-1}, e_{n-1}$ and $\tilde{\lambda}_{n-1}$ are defined and regular on $[0, T_{n-1}[\times[0, +\infty[$, with $T_{n-1} > 0$, then define

$$F_{n-1}(t, \tilde{x}, v) = (F_{1,n-1}; F_{2,n-1})(t, \tilde{x}, v) \tag{4.4}$$

where, following proposition 3.1:

$$F_{1,n-1}(t, \tilde{x}, v) = e^{\mu_{n-1}-\lambda_{n-1}} \frac{v}{\sqrt{1+v^2}} \tag{4.5}$$

$$\begin{cases} F_{2,n-1}(t, \tilde{x}, v) = -(\tilde{\lambda}_{n-1}\frac{\tilde{x}.v}{r} + e^{\mu_{n-1}-\lambda_{n-1}}\mu'_{n-1}\sqrt{1+v^2} - qe_{n-1}e^{\mu_{n-1}+\lambda_{n-1}})\frac{\tilde{x}}{r}, \text{ if } \tilde{x} \neq 0 \\ 0 \quad \tilde{x} = 0 \end{cases}$$

(4.6)

for $t \in [0, T_{n-1}[$ and $(\tilde{x}, v) \in \mathbb{R}^6$, denote by $Z_n(., t, z) = (X_n, V_n)(., t, \tilde{x}, v)$ the solution of the characteristic system

$$\dot{z} = F_{n-1}(s, z)$$

with $Z_n(t, t, z) = z$, and define

$$f_n(t, z) = \overset{\circ}{f}(Z_n(0, t, z)), \quad t \in [0, T_{n-1}[, \quad z \in \mathbb{R}^6,$$

i.e f_n satisfies the auxiliary Vlasov equation:

$$\frac{\partial f_n}{\partial t} + F_{1,n-1}.\frac{\partial f_n}{\partial \tilde{x}} + F_{2,n-1}.\frac{\partial f_n}{\partial v} = 0 \tag{4.7}$$

with $f_n(0) = \overset{\circ}{f}$, and:

$$\begin{cases} \rho_n(t, \tilde{x}) = \int_{\mathbb{R}^3} f_n(t, \tilde{x}, v)\sqrt{1+v^2}dv + \frac{1}{2}e^{2\lambda_{n-1}(t,\tilde{x})}e^2_{n-1}(t, \tilde{x}) \\ p_n(t, \tilde{x}) = \int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r}\right)^2 f_n(t, \tilde{x}, v)\frac{dv}{\sqrt{1+v^2}} - \frac{1}{2}e^{2\lambda_{n-1}(t,\tilde{x})}e^2_{n-1}(t, \tilde{x}) \\ k_n(t, \tilde{x}) = \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r}f_n(t, \tilde{x}, v)dv \end{cases} \tag{4.8}$$

$$m_n(t, r) = 4\pi \int_0^r s^2\rho_n(t, s)ds = \int_{|y|\leq r} \rho_n(t, y)dy \tag{4.9}$$

$$\begin{cases} N_n(t, \tilde{x}) = \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{\sqrt{1+v^2}}f_n(t, \tilde{x}, v)dv \\ M_n(t, \tilde{x}) = \int_{\mathbb{R}^3} f_n(t, \tilde{x}, v)dv. \end{cases} \tag{4.10}$$

Now, (3.7) can be use to define λ_n as long as the right hand side is positive. Thus we define

$$T_n := \sup\{t \in [0, T_{n-1}[| 2m_n(s, r) < r, r \geq 0, s \in [0, t]\} \tag{4.11}$$

and let

$$\begin{cases} e^{-2\lambda_n(t,r)} := 1 - \frac{2m_n(t,r)}{r} \\ \lambda_n(0, r) := \overset{\circ}{\lambda} \end{cases} \tag{4.12}$$

13

$$\mu'_n := e^{2\lambda_n(t,r)}\left(\frac{m_n(t,r)}{r^2} + 4\pi r p_n(t,r)\right) \tag{4.13}$$

$$\mu_n(t,r) := -\int_r^{+\infty} \mu'_n(t,s)ds \tag{4.14}$$

$$\tilde{\lambda}_n(t,r) := -4\pi r e^{(\lambda_n+\mu_n)(t,r)} k_n(t,r) \tag{4.15}$$

$$e_n(t,r) := \frac{q}{r^2}e^{-\lambda_n(t,r)}\int_0^r s^2 e^{\lambda_n(t,s)} M_n(t,s)ds. \tag{4.16}$$

We deduce from (4.12) that:

$$\lambda'_n = e^{2\lambda_n(t,s)}\left(-\frac{m_n(t,r)}{r^2} + 4\pi r\rho_n(t,r)\right). \tag{4.17}$$

We also use the Vlasov equation (3.1) and Gauss theorem to obtain the analogous conservation law given by (3.3):

$$\frac{\partial}{\partial t}\left(e^{\lambda_n}\int_{\mathbb{R}^3} f_n dv\right) = -\underset{\tilde{x}}{\operatorname{div}}\left(e^{\lambda_n+\mu_{n-1}-\lambda_{n-1}}\int_{\mathbb{R}^3}\frac{v}{\sqrt{1+v^2}}f_n dv\right)$$
$$+ (\dot{\lambda}_n - \tilde{\lambda}_{n-1})e^{\lambda_n}M_n$$
$$+ (\lambda'_n - \lambda'_{n-1})\frac{N_n}{r}e^{\lambda_n+\mu_{n-1}-\lambda_{n-1}}. \tag{4.17'}$$

So, multiplying (4.16) by e^{λ_n} and differentiating the obtained equation with respect to t, using (4.17') and Gauss theorem, we have:

$$\frac{\partial}{\partial t}(e^{\lambda_n}e_n) = -q\frac{N_n}{r}e^{\lambda_n+\mu_{n-1}-\lambda_{n-1}} + \frac{q}{4\pi r^2}\int_{|y|\leq r}(\dot{\lambda}_n - \tilde{\lambda}_{n-1})e^{\lambda_n}M_n dy$$
$$+ \frac{q}{4\pi r^3}\int_{|y|\leq r}(\lambda'_n - \lambda'_{n-1})e^{\lambda_n+\mu_{n-1}-\lambda_{n-1}}dy. \tag{4.18}$$

We now prove that all the above expression make sense.

Proposition 4.1 *For all $n \in \mathbb{N}$, the functions λ_n, μ_n, f_n, e_n, p_n, k_n, N_n, M_n, $\tilde{\lambda}_n$ are well defined and regular, $T_n > 0$, and $\mu_n + \lambda_n \leq 0$, $\lambda_n \geq 0$, $\mu_n \leq 0$.*

Proof: This assertion follows by induction, using proposition 3.1, proposition 3.2, and the construction of $\overset{\circ}{\lambda}$, $\overset{\circ}{e}$. The crucial step in this proof is to show that $T_n > 0$. To do so, we take $t \leq \max(1, \frac{T_{n-1}}{2})$ and obtain

$$\int_{\mathbb{R}^3} \rho_n(t,y)dy \leq C_n \tag{4.18'}$$

where C_n is a constant. How do we see the latter? In fact there are two terms in the left hand side of (4.18'); the first due to the compact support of $f_n(t)$ is

14

bounded, while the second can be written in polar coordinates and using the following formula

$$\int_0^s \tau^2 e^{\lambda_{n-1}} M_{n-1} d\tau = \frac{s^3}{3} e^{\lambda_{n-1}} M_{n-1} - \frac{1}{3} \int_0^s \tau^3 e^{\lambda_{n-1}} (\lambda'_{n-1} M_{n-1} + M'_{n-1}) d\tau$$

(4.18")

as:

$$\frac{1}{2} \int_{\mathbb{R}^3} e^{2\lambda_{n-1}} e_{n-1}^2 dy = \frac{2\pi}{3} q^2 \lim_{r \to +\infty} \int_0^r s e^{\lambda_{n-1}} M_{n-1} ds$$
$$- \frac{2\pi}{3} q^2 \lim_{r \to +\infty} \int_0^r \frac{1}{s^2} \int_0^s \tau^3 e^{\lambda_{n-1}} (\lambda'_{n-1} M_{n-1} + M'_{n-1}) d\tau ds.$$

(4.18"')

Now, since $M_{n-1}(t)$ and then $M'_{n-1}(t)$ are compactly supported, we can conclude that the left hand side of (4.18"') is bounded and then (4.18') holds as well. Next, choose $R > 0$ such that $\frac{C_n}{R} < \frac{1}{2}$, since $\frac{m_n}{r}$ is uniformly continuous on $[0, \max(1, \frac{T_{n-1}}{2}] \times [0, R]$ and $\frac{m_n(0,r)}{r} < \frac{1}{2}$ for $r > 0$, there exists $T' \in]0, \max(1, \frac{T_{n-1}}{2}]$ such that $\frac{m_n(t,r)}{r} < \frac{1}{2}$, for $t \in [0, T']$ and $r \in [0, R]$. Thus $0 < T' \le T_n$ and we have the desired result.

Note that the regularity of $\tilde{\lambda}_n$ and e_n follows from the identities:

$$\tilde{\lambda}'_n = \tilde{\lambda}_n(\mu'_n + \lambda'_n) - 4\pi e^{\mu_n + \lambda_n} k_n - 4\pi r e^{\mu_n + \lambda_n} k'_n$$
$$e'_n = q M_n - \lambda'_n e_n - \frac{2e_n}{r}$$

and the regularity of k_n. So proposition 4.1 is proved.

Now, to establish the convergence of iterates we prove in the following result the existence of some bounds on iterates which are uniform in n

Proposition 4.2 *The sequence of functions stated above is bounded.*

Proof: First of all, we define

$$P_n(t) = \sup\{|v| \,|\, (\tilde{x}, v) \in \text{supp} f_n(s), \quad 0 \le s \le t\} \qquad (4.19)$$

$$Q_n(t) = \sup\{e^{2\lambda_n(s,r)}, \quad r \ge 0, \quad 0 \le s \le t\}. \qquad (4.20)$$

Since $\| f_n(t) \|_{L^\infty} = \| \overset{\circ}{f} \|_{L^\infty}$ for $t \in [0, T_n[$, we obtain for all $n \in \mathbb{N}$, the estimates

$$\begin{cases} \| k_n(t) \|_{L^\infty}, \| N_n(t) \|_{L^\infty} \le C \| \overset{\circ}{f} \|_{L^\infty} (1 + P_n(t) + Q_n(t))^4 \\ \| M_n(t) \|_{L^\infty} \le C \| \overset{\circ}{f} \|_{L^\infty} (r_0 + t)(1 + P_n(t) + Q_n(t))^3 \end{cases} \qquad (4.21)$$

and by virtue of (4.16), and the fact that $\lambda_n \ge 0$, one has:

$$\| e_n(t) \|_{L^\infty} \le C Q_n^{\frac{1}{2}}(t) \| \overset{\circ}{f} \|_{L^\infty} (1 + P_n(t) + Q_n(t))^3 (r_0 + t)$$
$$\left| \frac{e_n(t,r)}{r} \right| \le C Q_n^{\frac{1}{2}}(t) \| \overset{\circ}{f} \|_{L^\infty} (1 + P_n(t) + Q_n(t))^3 \qquad (4.22)$$

15

Thus,

$$\| \rho_n(t) \|_{L^\infty}, \| p_n(t) \|_{L^\infty} \le C(1 + r_0 + t)^2 \| \overset{\circ}{f} \|_{L^\infty} (1+ \| \overset{\circ}{f} \|_{L^\infty})R_n(t) \quad (4.23)$$

where $C > 0$ denotes a constant which in the sequel may change its value from line to line and does not depend on n, t and $\overset{\circ}{f}$, and where

$$R_n(t) = (1 + P_{n-2}(t) + Q_{n-2}(t))^7(1 + P_{n-1}(t) + Q_{n-1}(t))^7$$
$$\times (1 + P_n(t) + Q_n(t))^{14}(1 + P_{n+1}(t) + Q_{n+1}(t))^7.$$

We combine the estimates above with (4.13) and (4.15) to obtain, since $\lambda_n + \mu_n \le 0$:

$$| e^{(\mu_n - \lambda_n)(t,r)}\mu'_n(t,r) |\le C(r_0 + t) \| \overset{\circ}{f} \|_{L^\infty} (1+ \| \overset{\circ}{f} \|_{L^\infty})R_n(t) \qquad (4.24)$$

$$| \tilde{\lambda}_n(t,r) |\le C(r_0 + t) \| \overset{\circ}{f} \|_{L^\infty} (1+ \| \overset{\circ}{f} \|_{L^\infty})R_n(t). \qquad (4.25)$$

Note that $r =| \tilde{x} |\le r_0 + t$ for $f_n(t,\tilde{x},v) \ne 0$. Next, we insert theses estimates into the characteristic system which yields:

$$| \dot{V}_{n+1}(t,0,z) |\le C(1 + r_0 + t)(1+ \| \overset{\circ}{f} \|_{L^\infty})^2 R_n(t) \qquad (4.26)$$

Integrating (4.26) on $[0,t]$, one has:

$$| V_{n+1}(t,0,z) |\le| v | + \int_0^t | \dot{V}_{n+1}(s,0,z) | \, ds.$$

Thus,

$$P_{n+1}(t) \le u_0 + C \| \overset{\circ}{f} \|_{L^\infty} (1+ \| \overset{\circ}{f} \|_{L^\infty}) \int_0^t (1 + r_0 + s)R_n(s)ds. \qquad (4.27)$$

Next, we look for an inequality for $Q_n(t)$. We can write, using (4.12):

$$\left| \frac{\partial}{\partial t}e^{2\lambda_{n+1}(t,r)} \right| \le 2Q_{n+1}^2(t)\left| \frac{\dot{m}_{n+1}(t,r)}{r} \right|, \qquad (4.27')$$

we see that we need an estimate for the time derivative of m_{n+1} in (4.9). We calculate $\dot{m}_{n+1}(t,r)$, use (4.7)(to express $\frac{\partial f_n}{\partial t}$) Gauss theorem, (4.18) and the Gronwall lemma to obtain the estimate; since $\lambda_n + \mu_n \le 0$:

$$2Q_{n+1}^2(t)\frac{| \dot{m}_{n+1}(t,r) |}{r} \le C \exp \left(C(1 + r_0 + t)^8(1+ \| \overset{\circ}{f} \|_{L^\infty})^3 \sup_{i \le n} R_i(t) \right). \qquad (4.28)$$

Next, we integrate $(4.27')$ on $[0,t]$ using (4.28) and obtain, with

$$q_0 = Q_{n+1}(0) = \sup\{e^{\overset{\circ}{\lambda}(r)}, r \ge 0\}$$

16

$$Q_{n+1}(t) \le q_0 + C \int_0^t \exp\left(C(1 + r_0 + s)^8 (1 + \| \overset{\circ}{f} \|_{L^\infty})^3 \sup_{i \le n} R_i(s) \right) ds. \quad (4.29)$$

Now, consider

$$\begin{cases} \tilde{P}_n(t) = \sup_{m \le n} P_m(t) \\ \tilde{Q}_n(t) = \sup_{m \le n} Q_m(t); \end{cases}$$

then $\tilde{P}_n, \tilde{Q}_n$ are increasing sequences and for all n one has $P_n \le \tilde{P}_n$, $Q_n \le \tilde{Q}_n$. Using the above expression of R_n, one deduces:

$$R_n(t) \le (1 + \tilde{P}_{n+1}(t) + \tilde{Q}_{n+1}(t))^{35}.$$

Now fix $n \in \mathbb{N}$ and write (4.27) and (4.29) for every m, where $m \le n$. Taking the supremum over $m \le n$, yields

$$\tilde{P}_{n+1}(t) + \tilde{Q}_{n+1}(t) \le u_0 + q_0$$
$$+ C \int_0^t \exp\left(C\Lambda(s)(1 + \tilde{P}_{n+1}(s) + \tilde{Q}_{n+1}(s))^{35} \right) ds,$$

where

$$\Lambda(s) := (1 + r_0 + s)^8 (1 + \| \overset{\circ}{f} \|_{L^\infty})^3$$

and by the Gronwall lemma, $\tilde{P}_{n+1}, \tilde{Q}_{n+1}$ and hence P_n, Q_n are bounded on the domain $[0, T^0]$, $T^0 \ge 0$, of the maximal solution z_0 of

$$z_0(t) = u_0 + q_0 + C \int_0^t \exp\left(C\Lambda(s)(1 + z_0(s))^{35} \right) ds \quad (4.30)$$

It follows that, $P_n(t) + Q_n(t) \le z_0(t)$, $n \in \mathbb{N}$, $t \in [0, T^0[\cap [0, T_n[$, and by definition $T_n \ge T^0$, $n \in \mathbb{N}$ and the proof is complete.

Now, in the following $C(t)$ denotes an increasing, continuous function on $[0, T^0[$ which depends on z_0, but not on n. From the estimates in proposition 4.2 we deduce:

$$\| \rho_n(t) \|_{L^\infty}, \| p_n(t) \|_{L^\infty}, \| k_n(t) \|_{L^\infty}, \| \lambda_n(t) \|_{L^\infty} \| N_n(t) \|_{L^\infty},$$
$$\| M_n(t) \|_{L^\infty}, \| \mu_n(t) \|_{L^\infty}, \| \tilde{\lambda}_n(t) \|_{L^\infty}, \| \mu'_n(t) \|_{L^\infty}, \| \lambda'_n(t) \|_{L^\infty},$$
$$\| e_n(t) \|_{L^\infty}, \| e'_n(t) \|_{L^\infty} \le C(t), \quad t \in [0, T^0[.$$

Next we need to know more about some bounds on certain derivatives. We do it by proving the following result:

Proposition 4.3 *There exists a unique nonnegative function $z_1 \in C^1$ defined on some interval $[0, T^1[$ such that:*

$$\| \partial_{\tilde{x}} f_n(t) \|_{L^\infty} \le z_1(t), \quad t \in [0, T^1[, n \in \mathbb{N}.$$

17

Proof: We have the following estimates:

$$
\begin{cases}
\parallel \tilde{\lambda}'_n(t) \parallel_{L^\infty} \leq C(t)(1+ \parallel k'_n(t) \parallel_{L^\infty}) \\
\parallel \mu''_n(t) \parallel_{L^\infty} \leq C(t)(1+ \parallel p'_n(t) \parallel_{L^\infty}) \\
\parallel \lambda''_n(t) \parallel_{L^\infty} \leq C(t)(1+ \parallel \rho'_n(t) \parallel_{L^\infty}
\end{cases}
\tag{4.31}
$$

and by the regularity of k_n, we have, using (4.9), Gauss theorem:

$$
\parallel k'_n(t) \parallel_{L^\infty} + \parallel M'_n(t) \parallel_{L^\infty} + \parallel N'_n(t) \parallel_{L^\infty} \leq C(t) \parallel \partial_{\tilde{x}} f_n(t) \parallel_{L^\infty}
$$
$$
\parallel \rho'_n(t) \parallel_{L^\infty}, \parallel p'_n(t) \parallel_{L^\infty} \leq C(t)(1+ \parallel \partial_{\tilde{x}} f_n(t) \parallel_{L^\infty})
$$

Next, the definition of f_n implies that

$$
\parallel \partial_{\tilde{x}} f_n(t) \parallel_{L^\infty} \leq \parallel \partial_{\tilde{x}} \overset{\circ}{f} \parallel_{L^\infty} \sup\{ | \partial z Z_n(0,t,z) |, \quad z \in \mathrm{supp} f_n(t)\}
\tag{4.32}
$$

and $\partial_z \dot{Z}_{n+1}(s,t,z) = \partial_z F_n(s, Z_{n+1}(s,t,z)).\partial_z Z_{n+1}(s,t,z)$. The derivative $\partial_z F_n(s,t,z)$ contains terms which are bounded by proposition 4.2, terms like $\frac{\tilde{\lambda}_n}{r}, \frac{\mu'_n}{r}, \frac{\lambda'_n}{r}$, e_n, e'_n and $\frac{e_n}{r}$ which are again bounded by proposition 4.2, and the terms μ''_n, $\tilde{\lambda}'_n$. Thus

$$
\sup\{ | \partial_z F_n(s, \partial_z F_n(s, \tilde{x}, , v) |; \tilde{x} \in \mathbb{R}^3 \mid v |\leq z_0(s)\} \leq C(s)(1+ \parallel \partial_{\tilde{x}} f_n(s) \parallel_{L^\infty})
$$

and

$$
| \partial_z \dot{Z}_{n+1}(s,t,z) |\leq C(s)(1+ \parallel \partial_{\tilde{x}} f_n(s) \parallel_{L^\infty}) | \partial_z Z_{n+1}(s,t,z) |
\tag{4.33}
$$

for any characteristics $Z_{n+1}(s,t,z)$ with $z \in \mathrm{supp} f_{n+1}(t)$, and for which therefore, by proposition 4.2, $| V_{n+1}(s,t,z) |\leq z_0(s)$. By the Gronwall lemma, one deduces from integration of (4.33) on $[s,t]$, since $Z_{n+1}(t,t,z) = z$:

$$
| \partial_z Z_{n+1}(s,t,z) |\leq \exp\left(\int_s^t C(\tau)(1+ \parallel \partial_{\tilde{x}} f_n(\tau) \parallel_{L^\infty}) d\tau \right)
$$

and combining this with (4.32), we obtain the inequality:

$$
\parallel \partial_{\tilde{x}} f_{n+1}(t) \parallel \leq \parallel \partial_z \overset{\circ}{f} \parallel_{L^\infty} \exp\left(\int_s^t C(s)(1+ \parallel \partial_{\tilde{x}} f_n(s) \parallel_{L^\infty}) ds \right).
\tag{4.34}
$$

Let z_1 be the maximal solution of

$$
z_1(t) = \parallel \partial_z \overset{\circ}{f} \parallel_{L^\infty} \exp\left(\int_s^t C(s)(1 + z_1(s)) ds \right)
\tag{4.35}
$$

which exists on some interval $[0, T^1[\subset [0, T^0[$; recall that $C(t) = C(t, z_0)$. Then, we have:

$$
\parallel \partial_{\tilde{x}} f_n(t) \parallel_{L^\infty} \leq z_1(t), \quad t \in [0, T^1[, \quad n \in \mathbb{N}
$$

and therefore the quantities $\tilde{\lambda}'_n$ and μ''_n, can also be estimated in terms of z_1 on the time interval $[0, T^1[$ uniformly in n. This completes the proof of proposition 4.3.

18

4.2 The convergence of iterates

Here we show that the above sequence of iterates which we constructed converges. We prove in the sequel this important result:

Proposition 4.4 *The sequence of iterates* $(f_n, \lambda_n, \mu_n, e_n)$ *converges.*

Proof: Let $\delta \in]0, T^1[$. By proposition 4.2 ,

$$\| k_{n+1}(t) - k_n(t) \|_{L^\infty}, \| N_{n+1}(t) - N_n(t) \|_{L^\infty},$$
$$\| M_{n+1}(t) - M_n(t) \|_{L^\infty} \leq C \| f_{n+1}(t) - f_n(t) \|_{L^\infty} . \tag{4.36}$$

Now, by the definition of e_n, one has; distinguishing the cases $r \leq r_0$ and $r \geq r_0$:

$$| e^{\lambda_{n+1}} e_{n+1} - e^{\lambda_n} e_n | (t, r) \leq C \| f_{n+1}(t) - f_n(t) \|_{L^\infty}$$
$$+ C \int_0^r | M_{n+1}(t, s) || e^{\lambda_{n+1}} - e^{\lambda_n} | (t, s) ds. \tag{4.37}$$

We find an estimate for $e^{\lambda_{n+1}} - e^{\lambda_n}$. Using the definition (4.12) of $e^{-2\lambda_n}$, we have

$$e^{\lambda_{n+1}} - e^{\lambda_n} = \frac{2}{r} e^{2\lambda_n + \lambda_{n+1}} \frac{m_{n+1} - m_n}{1 + e^{\lambda_n - \lambda_{n+1}}}$$

and since e_n and λ_n are bounded, we obtain:

$$| e^{\lambda_{n+1}} - e^{\lambda_n} | (t, r) \leq C \| f_{n+1}(t) - f_n(t) \|_{L^\infty}$$
$$+ C \int_0^r s | e^{\lambda_n} e_n - e^{\lambda_{n-1}} e_{n-1} | (t, s) ds. \tag{4.38}$$

Next, inserting (4.38) into (4.37) and distinguishing the cases $r \leq r_0$ and $r \geq r_0$, using permutation of variables in the double integral that appears inside the obtained inequality and applying lemma 3.2 to obtain:

$$| e^{\lambda_{n+1}} e_{n+1} - e^{\lambda_n} e_n | (t, r) \leq C \sum_{i=1}^n \| f_{i+1}(t) - f_i(t) \|_{L^\infty} + C \frac{C^n (r_0 + \delta)^n}{n!}. \tag{4.39}$$

Thus, from $-\lambda_n \leq 0$ and $\| e_n(t) \|_{L^\infty} \leq C$, we obtain:

$$\| e_{n+1}(t) - e_n(t) \|_{L^\infty}, \| \rho_{n+1}(t) - \rho_n(t) \|_{L^\infty},$$
$$\| p_{n+1}(t) - p_n(t) \|_{L^\infty} \leq C \sum_{i=1}^n \| f_{i+1}(t) - f_i(t) \|_{L^\infty} + C \sum_{i=n-1}^n \frac{C^i (r_0 + \delta)^i}{i!} \tag{4.40}$$

and we deduce also, since λ_n is bounded, the quantities

$$\| \mu_{n+1}(t) - \mu_n(t) \|_{L^\infty}, \| \mu'_{n+1}(t) - \mu'_n(t) \|_{L^\infty}, \| \tilde{\lambda}_{n+1}(t) - \tilde{\lambda}_n(t) \|_{L^\infty},$$
$$\| \lambda'_{n+1}(t) - \lambda'_n(t) \|_{L^\infty}, \| \lambda_{n+1}(t) - \lambda_n(t) \|_{L^\infty}, \| e'_{n+1}(t) - e'_n(t) \|_{L^\infty}$$

satisfy (4.40). Now,

$$\sup\{|\ F_{n+1} - F_n\ |\ (s, \tilde{x}, v)|\quad \tilde{x} \in \mathbb{R}^3, |\ v\ |\leq z_0(s)\}$$

satisfies (4.40) and by proposition 4.3,

$$\sup\{|\ \partial_z F_n(s, \tilde{x}, v)\ ||\quad \tilde{x} \in \mathbb{R}^3, |\ v\ |\leq z_0(s)\} \leq C$$

for $s \in [0, \delta]$, and the estimate of the difference of two iterates of characteristics gives, since $(\dot{Z}_{n+1} - \dot{Z}_n)(s, t, z) = (F_n - F_{n-1})(s, t, z)$:

$$|\ \dot{Z}_{n+1} - \dot{Z}_n\ |\ (s, t, z) \leq C\ |\ Z_{n+1} - Z_n\ |\ (s, t, z) + C\sum_{i=1}^{n}\ \|\ f_{i+1}(t) - f_i(t)\ \|_{L^\infty}$$

$$+ C\sum_{i=n-1}^{n}\frac{C^i(r_0 + \delta)^i}{i!}$$

$$(4.41)$$

for $z \in \mathrm{supp} f_{n+1}(t) \cup \mathrm{supp} f_n(t)$; note that $|\ Z_i\ |\ (s, t, z) \leq z_0(s)$, for $i = n, n+1$, and $s \in [0, \delta]$; i.e the characteristics run in the set on which we have bounded $\partial_z F_n$. Gronwall's lemma implies, after integrating (4.41)on $[0, t]$:

$$|\ Z_{n+1} - Z_n\ |\ (0, t, z) \leq C\delta\sum_{i=n-1}^{n}\frac{C^i(r_0 + \delta)^i}{i!} + C\sum_{i=1}^{n}\int_0^t\ \|\ f_{i+1}(s) - f_i(s)\ \|_{L^\infty}\ ds.$$

Thus, from

$$\|\ f_{n+1}(t) - f_n(t)\ \|_{L^\infty} \leq \|\ \partial_z \overset{\circ}{f}\ \|_{L^\infty} \sup\{|\ Z_{n+1} - Z_n\ |\ (0, t, z),$$
$$z \in \mathrm{supp} f_{n+1}(t) \cup \mathrm{supp} f_n(t)\}$$

we deduce, using once again Gronwall's lemma:

$$\|\ f_{n+1}(t) - f_n(t)\ \|_{L^\infty} \leq C\delta\sum_{i=n-1}^{n}\frac{C^i(r_0 + \delta)^i}{i!} + C\sum_{i=1}^{n-1}\int_0^t\ \|\ f_{i+1}(s) - f_i(s)\ \|_{L^\infty}\ ds.$$

Thus, by induction, we obtain the following estimate:

$$\|\ f_n(t) - f_{n-1}(t)\ \|_{L^\infty} \leq C\frac{C^n(1 + r_0 + \delta)^n}{n!}, \quad n \geq 1 \qquad (4.42)$$

where C depends on z_0 and not on n. Now, consider two integers m and n such that $m > n$. Then

$$\|\ f_m(t) - f_n(t)\ \|_{L^\infty} \leq C\sum_{i=m}^{n+1}\frac{C^i(1 + r_0 + \delta)^i}{i!}$$

and the right hand side of inequality above goes to zero as m and n go to infinity, since the series $\sum_{n=0}^{\infty}\frac{C^n(1+r_0+\delta)^n}{n!}$ converges. We conclude that $f_n(t)$ is a Cauchy sequence in the complete space L^∞, for all $t \in [0, \delta]$, and since all the differences which appear in (4.40) can be written in the form (4.42) such that the same holds for all sequences of functions that appear in (4.40) and others. So, the proof of proposition 4.4 is now complete.

4.3 The local existence and uniqueness theorem

In this section, we use lemma 3.1 to show that the limit obtained in proposition 4.4 is regular and thus is a solution of the auxiliary system under consideration. We replace λ, μ, $\tilde{\lambda}$, e in that lemma by λ_n, μ_n, $\tilde{\lambda}_n$, e_n and choose an arbitrary compact subinterval $[0, \delta] \subset [0, T^1[$ and $U > 0$. Here the essential result to proved is the following:

Theorem 4.1 (local existence and uniqueness) *The limit* (f, λ, μ, e) *of sequence* $(f_n, \lambda_n, \mu_n, e_n)$ *is a unique regular solution of the initial value problem under consideration with* $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{e})$.

Proof: The following bounds will be essential:

$$| a_{n,i}(s, \tilde{x}, v) | \leq C, n \in \mathbb{N}, i = 1, 2, 3, 4, (s, \tilde{x}, v) \in [0, \delta] \times \mathbb{R}^3 \times B(U) \quad (4.43)$$

$$| \partial_z a_{n,i}(s, \tilde{x}, v) | \leq C, n \in \mathbb{N}, i = 1, 2, 3, 4, (s, \tilde{x}, v) \in [0, \delta] \times \mathbb{R}^3 \setminus \{0\} \times B(U)$$
$$(4.44)$$

where $B(U)$ is the open ball of $\mathbb{R}^3$ with center O and with radius U.

The bounds for $a_{n,1}$, $a_{n,2}$ and $a_{n,4}$ follow immediately from those established in proposition 4.2 and

$$\frac{m_n(t, r)}{r^3} \leq 4\pi \parallel \rho_n(t) \parallel_{L^\infty}$$

we deduce the bound on $a_{n,3}$. Obviously, the derivatives of $a_{n,i}$ w.r.t v exist and are bounded on the set indicated above for $i = 1, 2, 3, 4$. The derivatives of $a_{n,1}$, $a_{n,2}$ and $a_{n,4}$ w.r.t $\tilde{x}$ also exist and are bounded, since the term μ_n'', λ_n'' and $\tilde{\lambda}_n'$ which appear in these derivatives in addition to (4.31) were established in proposition 4.3. The only qualitatively new terms which appear in $\partial_{\tilde{x}} a_{n,3}$ are

$$\left(\frac{\mu_n'}{r}\right)'; \quad \left(\frac{\lambda_n'}{r}\right)'; \quad \left(\frac{\tilde{\lambda}_n}{r}\right)'; \quad e_n''; \quad \frac{e_n'}{r} - \frac{e_n}{r^2}.$$

The third term of these are bounded by proposition 4.3. In the two first terms, the critical term is $\left(\frac{m_n(t,r)}{r^3}\right)'$, but for $r > 0$,

$$\left| \left(\frac{m_n(t, r)}{r^3}\right)' \right| \leq 7\pi \parallel \rho_n'(t) \parallel_{L^\infty} .$$

We now look for bounds of the two last terms. To do so we calculate e_n'' using (3.13) and (4.18") to obtain:

$$e_n''(t, r) = -\frac{2q}{r^4} e^{-\lambda_n(t,r)} \int_0^r s^3 e^{\lambda_n(t,s)} (\lambda_n' M_n + M_n')(t, s) ds + \frac{4}{r} e_n(t, r) \lambda_n'(t, r)$$
$$+ (\lambda_n'^2 - e_n \lambda_n'' - q\lambda_n M_n + qM_n')(t, r)$$

21

from which we deduce the following bound of e_n'':

$$\| e_n''(t) \|_{L^\infty} \leq C(t)(1+ \| \rho_n'(t) \|_{L^\infty} + \| p_n'(t) \|_{L^\infty} + \| M_n'(t) \|_{L^\infty}).$$

Using once again (3.13) and (4.18") we obtain:

$$\left(\frac{e_n'}{r} - \frac{e_n}{r^2} \right)(t,r) = \frac{2q}{3r^4} e^{-\lambda_n(t,r)} \int_0^r s^3 e^{\lambda_n(t,s)} (\lambda_n' M_n + M_n')(t,s)ds - \frac{1}{r} \lambda_n(t,r) e_n(t,r),$$

from which we deduce the following bound of $\frac{e_n'}{r} - \frac{e_n}{r^2}$:

$$\left| \frac{e_n'}{r} - \frac{e_n}{r^2} \right|(t,r) \leq C(t)(1+ \| M_n'(t) \|_{L^\infty}),$$

and the existence of the $\partial_z a_{n,i}$ bound's is proved. Now, the convergence established in proposition 4.4 shows that $| a_{n,i} - a_{m,i} | (s,\tilde{x},v) \underset{n,m\to\infty}{\to} 0$, for $i = 1,2,3,4$ and uniformly on $[0,\delta] \times \mathbb{R}^3 \times B(U)$. Therefore, the crucial term in the present argument is

$$H_n = e^{-2\lambda_n} \left(\mu_n'' + (\mu_n' - \lambda_n') \left(\mu_n' + \frac{1}{r} \right) \right) - e^{-2\mu_n} \left(\dot{\tilde{\lambda}}_n + \tilde{\lambda}_n (\dot{\lambda}_n - \dot{\mu}_n) \right)$$

which appears in $a_{n,5}$. We use the same calculations that we did when proving proposition 4.3, using Gauss theorem, the Vlasov equation and proposition 4.4 to obtain that: $H_n - 4\pi \bar{q}_n \underset{n\to\infty}{\to} 0$; $\dot{\lambda}_n - \tilde{\lambda}_{n-1} \underset{n\to\infty}{\to} 0$, uniformly on $[0,\delta] \times [0,+\infty[$, where $\bar{q}_n$ is obtained from (2.16) by replacing f, λ, e, by f_n, λ_n, e_n respectively.

The above estimates on the coefficients in lemma 3.1 show that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n,m > N$ we have the different inequalities:

$$| \dot{\xi}_{n,j}(s) - \dot{\xi}_{m,j}(s) | \leq \varepsilon + C(| \xi_{n,j}(s) - \xi_{m,j}(s) | + | \eta_{n,j}(s) - \eta_{m,j}(s) |)$$
$$| \dot{\eta}_{n,j}(s) - \dot{\eta}_{m,j}(s) | \leq \varepsilon + C(| \xi_{n,j}(s) - \xi_{m,j}(s) | + | \eta_{n,j}(s) - \eta_{m,j}(s) |)$$

The Gronwall lemma now shows that $(\xi_{n,j})$ and $(\eta_{n,j})$ are the Cauchy sequences and thus also $(\partial_{z_j} X_n(s,t,z))$ and $(\partial_{z_j} V_n(s,t,z)))$ are the Cauchy sequences locally uniformly on $([0,T^1[)^2 \times \mathbb{R}^6$. Thus $Z_n(s,t,.) \in C^1(\mathbb{R}^6)$ for $s,t \in [0,T^1[$, $f(t) \in C_c^1(\mathbb{R}^6)$ for $t \in [0,T^1[$, and we deduce that $\rho(t), p(t) \in C_c^1(\mathbb{R}^3)$, $M(t) \in C_c^1(\mathbb{R}^3)$, $N(t) \in C_c^1(\mathbb{R}^3)$, and $k(t) \in C^1(\mathbb{R}^3 \setminus \{0\}) \cap C^1([0,+\infty[)$. The right hand side of the characteristic system is therefore continuously differentiable in z, and $Z(0,t,z)$ is differentiable also w.r.t t, thus $f \in C^1([0,T^1[\times\mathbb{R}^6)$ and $(\lambda, \mu, \tilde{\lambda}, e)$ is a regular solution of the auxiliary system. Now we can check if that solution takes the initial value $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{e})$ at $t = 0$. We established before that the convergence of iterates is uniform on some interval $[0,\delta]$. So we can deduce:

$$\begin{cases} f_n(t) \to f(t) \\ \lambda_n(t) \to \lambda(t) \\ \mu_n(t) \to \mu(t) \\ e_n(t) \to e(t) \end{cases} \quad \text{for all } t \in [0,\delta].$$

22

In particular this holds for $t = 0$. But by the construction of f_n and λ_n and separation of L^∞ one has immediately:

$$f(0) = \overset{\circ}{f}; \quad \lambda(0) = \overset{\circ}{\lambda}.$$

Since $\overset{\circ}{e}$ is a regular solution of constraint equation (3.19) we obtain, taking (3.13) at $t = 0$: $e(0) = \overset{\circ}{e}$ and the result for μ follows by using equations (2.8) and (3.18). We end the proof of theorem 4.1 by showing uniqueness.

Assume that we have two regular solutions $(\lambda_f, \mu_f, f, e_f)$, $(\lambda_g, \mu_g, g, e_g)$, with $\lambda_f(0) = \lambda_g(0)$, $\mu_f(0) = \mu_g(0)$, $f(0) = g(0)$, $e_f(0) = e_g(0)$. The estimates, which we applied to the difference of two consecutive iterates in proposition 4.4 can be applied in analogous fashion to the difference of f and g to obtain

$$\| f(t) - g(t) \|_{L^\infty} \leq C \int_0^t \| f(s) - g(s) \|_{L^\infty} \, ds$$

and using the Gronwall lemma, one concludes that $f(t) = g(t)$, and then $\lambda_f(t) = \lambda_g(t)$; $\mu_f(t) = \mu_g(t)$, $e_f(t) = e_g(t)$ as long as both solutions exists.

5 The continuation criterion of solutions

Here we establish the continuation criterion for local solutions which may allow us to extend that solutions for a large time t.

Theorem 5.1 (Continuation criterion) *Let (f, λ, μ, e) be a unique regular solution of the initial value problem under consideration with $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{e})$ defined on a maximal interval $I \subset \mathbb{R}$ of existence which is open and contains 0. If*

$$\sup\{| v | \mid (t, \tilde{x}, v) \in \mathrm{supp} f, \, t \geq 0\} < +\infty$$

then $\sup I = +\infty$, if

$$\sup\{| v | \mid (t, \tilde{x}, v) \in \mathrm{supp} f, \, t \leq 0\} < +\infty$$

then $\inf I = -\infty$

Proof: Let $[0, T[$ be the right maximal interval of existence of a regular solution (f, λ, μ, e), and assume that

$$P* = \sup\{| v | \mid \quad (t, \tilde{x}, v) \in \mathrm{supp} f\} < \infty$$

and $T < \infty$. We will show that under this assumption we can extend the solution beyond T, which is a contradiction. Take any $t_0 \in [0, T[$. Then the above proof shows that we obtain a solution $\bar{f}$ with initial value $\bar{f}(t_0) = f(t_0)$ on the common existence interval of the solution of

$$z_0(t) = U_0 + Q_0 + C \int_{t_0}^t \exp\left(C(1 + r_0 + s)^8 (1 + \| f(t_0) \|_{L^\infty})^3 (1 + z_0(s))^{35}\right) ds$$

$$z_1(t) = \| \partial_z f(t_0) \|_{L^\infty} \exp\left(\int_{t_0}^t C(s)(1 + z_1(s)) ds\right)$$

where $C(s)$ is a function which depends on z_0, and

$$U_0 = \sup\{|\, v\,|\,|\,(\tilde{x}, v) \in \mathrm{supp} f(t_0)\} < P*$$
$$R_0 = \sup\{|\, \tilde{x}\,|\,|\,(\tilde{x}, v) \in \mathrm{supp} f(t_0)\} < r_0 + T$$
$$Q_0 = \sup\{e^{2\lambda(t_0, r)}, \quad r \geq 0\}.$$

By proposition 4.4, $\dot{\lambda} = \tilde{\lambda} = -4\pi r e^{\lambda + \mu} k$, and thus $\|\, \dot{\lambda}\,\|_{L^\infty} \leq C$, $t \in [0, T[$, which implies the estimate

$$Q_0 \leq Q* = \sup\{e^{2\lambda(t, r)}, \quad t \in [0, T[, \quad r \geq 0\}$$

$|\, \partial_z Z(0, t, z)\,| \leq C$, for $z \in \mathrm{supp} f(t)$ and $t \in [0, T[$, since all coefficients in lemma 3.1 are bounded along the characteristics in $\mathrm{supp} f$; for the coefficient a_3 we observe that due to (2.9), $H = 4\pi\bar{q}$, where H denotes the left hand side of (2.9), and $\bar{q}$ is bounded due to the bound on $\mathrm{supp} f(t, \tilde{x}, .)$. Thus

$$\|\, \partial_z f(t_0)\,\|_{L^\infty} \leq \sup\{\|\, \partial_z f(t)\,\|_{L^\infty}\,|, \quad t \in [0, T[\} < +\infty.$$

These estimates imply that there exists $\delta > 0$, independent of t_0, such that (z_0, z_1) and thus also the solution $\bar{f}$, exists on the interval $[t_0, t_0 + \delta]$. For t_0 close enough to T this solution extends the solution f beyond T, which is a contradiction. Thus if $P* < \infty$ then $T = +\infty$ and this ends the proof of theorem 4.2. Using theorem 4.1 and theorem 5.1 we can prove the following essential result of this section:

Theorem 5.2 (local existence, continuation criterion) *Let $\overset{\circ}{f} \in C^\infty(\mathbb{R}^6)$ be nonnegative, compactly supported and spherically symmetric such that (4.1) be satisfied. Let $\overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{e} \in C^\infty(\mathbb{R}^3)$ be a regular solution of (3.17), (3.18) and (3.19). Then there exists a unique regular solution (λ, μ, f, e) of the spatial asymptotically flat spherically symmetric Einstein-Vlasov-Maxwell system with $(\overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{f}, \overset{\circ}{e})$ on a maximal interval $I \subset \mathbb{R}$ of existence which contains 0. If*

$$\sup\{|\, v\,|\,|(t, \tilde{x}, v) \in \mathrm{supp} f, \quad t \geq 0\} < +\infty$$

then $\sup I = +\infty$, if

$$\sup\{|\, v\,|\,|(t, \tilde{x}, v) \in \mathrm{supp} f, \quad t \leq 0\} < +\infty$$

then $\inf I = -\infty$.

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